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Conformally Invariant Gravitational Waves in a Modified Gravitational Theory

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An attempt is made to obtain a conformally invariant gravitational wave equation in an isotropic background universe by modifying the Einstein field equation through a correction term proposed in the Hilbert Lagrangian in the form of a series of finite terms in R ($\equiv g_{ik}R^{ik}$). It is shown that only those waves which are described by Bessel function $J_0(m\eta)$ in curved background can be transformed as classically periodic waves in flat background [without restricting the scale factor $a(\eta)$].

1. INTRODUCTION

It has been known already for a long time that some basic equations of theoretical physics, among them the equations for massless fields, are invariant with respect to the group C_g of conformal mapping (Fulton et al., 1962), the replacement of $g_{\mu\nu}$ and field variable $\phi_{\alpha\beta\cdots\nu}$ according to the rule $\bar{g}_{\mu\nu}=e^{-2\sigma}g_{\mu\nu}$, $\bar{\phi}_{\alpha\beta\cdots\nu}=e^{-\sigma(s+1)}\phi_{\alpha\beta\cdots\nu}$, where s is the spin of the field. For instance, the field equations for massless fields (with the well-known exception of the scalar, for which $\Box\phi+(R/6)\phi$ is conformally invariant, and not $\Box\phi=0$), remains unchanged under C_g. It is important to notice that the field variables transform with different powers of the conformal factor $\bar{e}^{2\sigma}$, depending on the spin of the field. For the scalar field, $\bar{\phi}=e^{-\sigma}\phi$, and for the Maxwell field equations ($s=1$), $\bar{A}_\alpha=A_\alpha$ or $\bar{F}_{\alpha\beta}=F_{\alpha\beta}$. In the case of the gravitational field ($s=2$), the conformal invariance is usually referred to the vacuum Bianchi identities with the Weyl tensor being the appropriate conformally invariant quantity.

Conformal symmetry of the field equations with respect to C_g is important from the physical point of view since it restricts the coupling of

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the physical system to the external gravitational field. For example, in electromagnetism it allows minimal coupling of the form $(\partial_\mu - eA_\mu)$ but not Pauli coupling of the form $\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi$. The role of conformal invariance in the context of quantum field theory in curved space-time has been emphasized many times (Parker, 1969; Zel'dovich and Novikov, 1975).

Now, the Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \tag{1}$$

derivable from the Hilbert action

$$A = \int R\sqrt{-g}\, d^4x + \kappa \sum_i \int m_i\, ds \tag{2}$$

predict the existence of gravitational waves in weak field approximation.

Weak gravitational waves belong to the class of weak gravitational fields, which can be regarded as given or imbedded in a flat background like other physical fields. Therefore, the metric everywhere in the considered region of space-time is taken to differ little from the Minkowski metric $\eta_{\mu\nu}$, that is, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$. This is not a conformally invariant transformation.

The vacuum Einstein equations reduce to the wave equation

$$h_{\mu\nu;\alpha}^{;\alpha} - 2 \overset{(0)}{R}_{\alpha\mu\nu\beta} h^{\alpha\beta} = 0 \tag{3}$$

in linear approximation. Here the correction $h_{\mu\nu}$ satisfies the gauge conditions $h_{00} = h_{0\alpha} = 0$, and $\overset{(0)}{R}_{\alpha\mu\nu\beta}$ is the Riemann tensor corresponding to background metric $\overset{(0)}{g}_{\mu\nu}$.

The gravitational field of a nonstationary isotropic universe is described by the metric

$$ds^2 = -g_{\mu\nu}\, dx^\mu\, dx^\nu = a^2(\eta)\left(d\eta^2 - dx^2 - dy^2 - dz^2\right) \tag{4}$$

(written in the conformally flat form) where η is related to the cosmic time t by the relation $c\, dt = a(\eta)\, d\eta$. Weak gravitational waves on the background of this metric are given by (Grishchuk, 1975, 1977)

$$h_i^{k\prime\prime} + 2\frac{a'}{a} h_i^{k\prime} + a^2 g^{lm} h^k_{i,l,m} = 0 \tag{5}$$

where the prime denotes the derivative with respect to η and the comma denotes the derivative with respect to spatial coordinates. Further, the wave correction to the metric can be represented in the form of a sum of terms (Lifshitz, 1946)

$$h_i^k = \frac{\mu}{a} G_i^k \tag{6}$$

where G_i^k is a tensor eigenfunction numbered m so the Laplace operator formed from the metric $dl^2 = dx^2 + dy^2 + dz^2$. Then, from (5) we obtain

$$\mu'' + \mu(m^2 - a''/a) = 0 \tag{7}$$

The effective potential a''/a in (7), distinguishes this equation from the ordinary wave equations in the Minkowski world. The fact that $a''/a \not\equiv 0$ (save for $a = \text{const}$, and $a = a_0\eta$) is a manifestation of the so-called conformal noninvariance of gravitational wave equations.

The analogy between electromagnetism and gravitation begins with the inverse square laws of Coulomb and Newton, but does not hold in equation (7). Therefore it is interesting to investigate whether the equation (7) can be put on a par with electromagnetic waves so far as their conformal invariance is concerned.

That is the purpose of this paper.

Since the gravitational waves are an inescapable consequence of the Einstein theory, it is necessary to alter the Einstein field equations in order to alter the gravitational wave equations. Or, to obtain a conformally invariant Hilbert Lagrangian without associating it with any field other than gravitation. For instance, many authors have suggested replacing the Hilbert Lagrangian (not conformally flat) with the Lagrangian $[g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + (R/6)\phi^2]$, which is conformally invariant. But this involves an extra massless scalar field which leads to the consequences like variant gravitational constant.

Here we attempt to achieve conformal invariance without introducing an extra field.

2. MODIFIED FIELD EQUATIONS

Over the years alternative theories have been postulated. For example, Weyl (1922) suggested the invariant R^2 to make the field action scale invariant (and to unify gravitation with electromagnetism), and another attractive alternative suggested is $R^{3/2}$ so that the coupling constant in matter Lagrangian is dimensionless. Breizman et al. (1971) studied the behavior of homogeneous isotropic universes whose underlying Lagrangian density depends on R^n (here n is a numerical constant), and Nariai (1973) considered this action in studying the problem of gravitational instability in an expanding universe. Also, quantum considerations led people to consider invariants like $R_{ik}R^{ik}$, R^2, and their combinations.

Weyl's theory (a conformally invariant theory) using R^2 has a bad long-range behavior as it does not reduce to Newtonian theory in the linearized limit. So we think the above combinations cannot serve the purpose. We want something to nullify the manifestation of gravitation, as evident from equation (7), without a special choice of the scale factor $a(\eta)$. So we propose to consider the Lagrangian density in the form (Pandey, 1978)

$$\mathfrak{L}_g = \sqrt{-g}\left[R - \sum_{n=2}^{N} C_n \frac{(l^2R)^n}{6l^2}\right] \tag{8}$$

where C_n are arbitrary coefficients corresponding to n, and l is a characteristic length. The characteristic radii of curvature of the background world are to be large compared with the gravitational wavelength.

Usually (see for example, Sokolov, 1976, who considers higher-order terms in R) one writes different coupling constants for each term; for instance, $\alpha R + \beta R^2 + \gamma R_{ik}R^{ik}$. This leads to confusion in analyzing the predictions of the theory. But here only one coupling constant (as in the Hilbert case) enters all the terms. The arbitrary coefficients C_n are dimensionless, and are introduced to cancel out the gravitational manifestation. In other words, C_n are to be determined so as to reduce to zero the additional potential in the gravitational wave equations without imposing any restriction on the scale factor $a(\eta)$, when we change the background metric from curved space-time to flat space-time to obtain a conformally invariant gravitational wave equation.

An application of the variational principle to the action

$$A = \int \left((\mathfrak{L}_g/\kappa) + \mathfrak{L}_s\right) d^4x$$

where $\mathfrak{L}_g$ is given by (8), and $\mathfrak{L}_s$ stands for the source Lagrangian density, gives the equations of the "modified field theory" as

$$\begin{aligned} G_{\mu\nu} - \sum_{n=2}^{N} \frac{nC_n}{6}(l^2R)^{n-1} \\ \times \Big[R_{\mu\nu} - \frac{1}{2n} g_{\mu\nu}R - (n-1)\left(R_{;\mu;\nu} - g_{\mu\nu}\Box R\right)/R \\ -(n-1)(n-2)\left(R_{;\mu}R_{;\nu} - g_{\mu\nu}R_{;\alpha}R^{;\alpha}\right)/R^2\Big] = \kappa T_{\mu\nu} \end{aligned} \tag{9}$$

Here

$$T_{\mu\nu} = \sqrt{-g}\,\frac{\delta \mathfrak{L}_s}{\delta g^{\mu\nu}} \tag{10}$$

stands for the energy momentum tensor responsible for the production of the gravitational potential $g_{\mu\nu}$. It can easily be seen that

$$T^{\nu}_{\mu;\,\nu} = 0 \tag{11}$$

holds for (9), as in general relativity.

It is evident that modified field equations (9) reduce to Einstein field equations (1) in the absence of the correction term $\Sigma_{n=2}^{N} C_n (l^2 R)^n / 6l^2$ in (8). This term has resulted in appearance of terms involving $\Sigma_{n=2}^{N} n C_n (l^2 R)^{n-1}/6$, in equation (9).

3. GRAVITATIONAL WAVES

As in the case of Einstein, the gravitational wave equations for the field equations (9) of the modified field theory can be obtained under weak-field approximation. The wave correction $h_{\mu\nu}$ satisfies as usual the gauge conditions ($h_{00} = h_{0\alpha} = 0$), and is expressed as the sum of terms like $h_i^k = \nu(\eta) G_i^k$ (Lifshitz, 1946). Then, we obtain in the modified theory the equation for ν as

$$\nu'' + \left(\frac{2a'}{a} + \frac{A'}{A} \right) \nu^{1} + m^2 \nu = 0 \tag{12}$$

where

$$A = 1 - \sum_{n=2}^{N} \frac{n C_n}{6} \left(\frac{l}{a} \right)^{2(n-1)} \left(\frac{6a''}{a} \right)^{n-1} \tag{13}$$

The number m indicates the spatial periodicity of the wave. Thus, equation (12) is the gravitational wave equation on the background metric (4) corresponding to the field equations (9). The equation (12) differs from the corresponding one in Einstein's theory in having an additional term A'/A in the coefficient of ν'. This additional term is the result of modification of Einstein theory, based on the Lagrangian density (8). However, this term is of small magnitude as the quantities (l/a) and (a''/a) in the expression (13) of A are obviously less than unity.

Further, it is easy to see that in case of a flat 3-space, any wave corresponding to any $m \neq 0$, is of very short wavelength as the radius of curvature of the background space is infinite. Now, in the following section, we study how this wave equation (12) transforms itself as we go from curved space-time (4) to Minkowski space-time.

4. CONFORMAL INVARIANCE

The wave equation (12) depends on the scale factor $a(\eta)$ and also on the arbitrary coefficients C_n. The former, however, is determined by the matter filling the universe, and by its equation of state. As mentioned in the above, we do not prescribe any restriction on $a(\eta)$, and therefore, make use of arbitrary coefficients C_n in nullifying the additional potential in the gravitational wave equation (12).

Consider the transformation

$$\mu = a\nu\sqrt{A} \tag{14}$$

This reduces (12) to

$$\mu'' + \mu\left[m^2 - \frac{\left(a\sqrt{A}\right)''}{a\sqrt{A}}\right] = 0 \tag{15}$$

Equation (15) is of the form of Schrödinger equation. Therefore, $|\mu|^2$ can be interpreted as being proportional to the energy density of the wave. However, it should be noted that the correction introduced in the Hilbert Lagrangian to modify the Einstein theory, has appeared in the wave equation (15) in the form of an effective potential given by

$$U(\eta) \equiv \frac{\left(a\sqrt{A}\right)''}{a\sqrt{A}} = \frac{a''}{a} + \frac{a'A'}{aA} + \frac{A''}{2A} - \frac{A'^2}{4A^2} \tag{16}$$

The first term of (16) is as in the case of Einstein field equations, and the other three terms are the consequence of modification of Einstein theory. We therefore have the possibility of nullifying the first term of (16) by its remaining three terms without imposing any restriction on the scale factor $a(\eta)$ through the proper choice of the arbitrary coefficients C_n. If so, $U(\eta) = 0$, and (15) then reduces to the usual Schrödinger equation in a flat background. Thus, demanding $U(\eta) \to 0$, at any given value of η, say, η_0 (η_0

can be taken at the start of the expansion of the universe), we obtain

$$k(\eta-\eta_0)=f(\eta_0)+f'(\eta_0)(\eta-\eta_0)+\frac{f''(\eta_0)}{\underline{|2}}(\eta-\eta_0)^2+\cdots \tag{17}$$

where k is a constant, and $f(\eta)\equiv a^2A$.

Comparing the coefficients of various powers of $(\eta-\eta_0)$ in (17), the values of the arbitrary coefficients C_n can be obtained by solving N simultaneous equations. These values of C_n are "suitable," and when used in the field equations (9), will result in eliminating $U(\eta)$ in equation (15). Further it should be noted that in the modified theory, the amplitude of the gravitational wave will be $\sqrt{A}$ times the amplitude of the gravitational wave in the Einstein theory, that is,

$$\nu^*=\nu\sqrt{A} \tag{18}$$

The first and second terms on the right-hand side of equation (17) correspond to two special cases of the scale factor, namely, $a=\text{const}$, and $a=a_0\eta$. In these two cases one does get a conformally invariant gravitational wave equation even from the Einstein field equations. The former corresponds to a flat background metric, and the latter to a universe in which the matter is filled with the equation of state $p=\epsilon/3$ (that is, near singularity in Friedmann cosmological model). The higher-order terms in (17) are due to the correction introduced in the Hilbert Lagrangian density.

Now we consider the impact of reducing $U(\eta)=0$, on the gravitational wave equation (12), which are obtained in linear approximation of the field equations (9) in an isotropic background universe (4). We find that

$$2a'/a+A'/A=1/(\eta-\eta_0) \tag{19}$$

Using (19) in (12), we obtain a Bessel-type equation

$$(\eta-\eta_0)^2\nu''+(\eta-\eta_0)\nu'+m^2(\eta-\eta_0)^2\nu=0 \tag{20}$$

The equation (20) throws light on the shape of gravitational waves, and also on their variations in the modified theory when the curved background [where it is described by $J_0(m\eta)$, which is a solution of (20)] is changed to a flat background metric (where it is described by the usual periodic wave) or vice versa.

5. DISCUSSION

It seems that one is left with two options, though each of them looks interesting. One of them is to agree that (14) is valid, and the other is its contrary. In the former case, we obtain the modified gravitational wave equations conformally invariant as the electromagnetic wave equations. As the conformal invariance preserves angle and phase (only lengths are altered), helicity should be preserved. And this is what we achieve in (14) through (18). Thus, we do not violate the spin ($s = 2$) of gravitation in a flat background; instead we modify slightly the amplitude of the gravitational waves to ν^*. As a consequence of this the structure of the gravitational waves in a curved background (4) is modified to a wave represented by the Bessel function $J_0(m\eta)$, from the usually classical periodic wave in a flat background. As such the modification in the amplitude of the wave in this modified field theory seems quite natural.

On the other hand, the latter case leads to the fact that any modification of Einstein theory based on scalar $R \equiv g_{ik}R^{ik}$, in its linearized version cannot yield a conformally invariant gravitational wave equation in an isotropic background universe. This means that on both classical and quantum levels, gravitons behave drastically differently from other massless particles. For instance, unlike photons, gravitons are coupled to one another. Classically, in a nonstationary isotropic gravitational field, particularly in the strong gravitational field of the early universe, gravitational waves can be amplified (because in this case they are not conformally invariant), and hence gravitons can be created (in the quantum sense).

Moreover, the field equation of the modified theory (9) involves terms like $\Box R/R$ and $R_{;i}R^{;i}/R^2$, which are obtained in the quantum theory of gravitation owing to renormalization of the stress tensor. Anomalies in the stress tensor are also known to involve such terms. Thus equation (9) may be a more natural choice for considering quantum gravitational processes in the early universe.

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